LOWER BOUNDS FOR MAHLER-TYPE MEASURES OF POLYNOMIALS

ARTŪRAS DUBICKAS AND IGOR PRITSKER

Abstract. In this note we consider various generalizations of the classical Mahler measure $M(P) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log|P(e^{i\theta})\,d\theta\right)$ for complex polynomials P, and prove sharp lower bounds for them. For example, we show that for any monic polynomial $P \in \mathbb{C}[z]$ satisfying $|P(0)| = 1$ the quantity $M_0(P) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\max(0, \log|P(e^{i\theta})|)\,d\theta\right)$ is greater than or equal to $M(1+z_1+z_2) = 1.381356\ldots$. This inequality is best possible, with equality being attained for all $P(z) = z^n + c$ with $n \in \mathbb{N}$ and $|c| = 1$.

1. Mahler measure and its relatives

The classical Mahler measure of a polynomial $P \in \mathbb{C}[z]$ is defined by

$$M(P) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log|P(e^{i\theta})|\,d\theta\right). \tag{1.1}$$

It is also known as the geometric mean, and an application of Jensen's formula for

$$P(z) = a_n\prod_{j=1}^{n}(z - z_j) \in \mathbb{C}[z],\ a_n \neq 0,$$

immediately gives that

$$M(P) = |a_n|\prod_{j=1}^{n}\max(1, |z_j|). \tag{1.2}$$

This direct connection with the roots of P explains why the Mahler measure plays an important role in various problems of number theory, see, e.g., [12] and [18]. For a polynomial in several variables $P \in \mathbb{C}[z_1, \ldots, z_d]$ its Mahler measure is defined by

$$M(P) = \exp\left(\frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi}\log|P(e^{i\theta_1}, \ldots, e^{i\theta_d})|\,d\theta_1\ldots d\theta_d\right),$$

but there is no expression similar to that in (1.2). See, e.g., [7] for the description of some problems and many references related to the multidimensional Mahler measure.

Another useful measure (height) of polynomials is given by

$$M_0(P) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log^+|P(e^{i\theta})|\,d\theta\right),$$

where we use the standard notation $\log^+|P(e^{i\theta})| = \max(0, \log|P(e^{i\theta})|)$ for $P \in \mathbb{C}[z]$. In fact, $M_0(P)$ is equal to the Mahler measure of the two variable polynomial $P(z_1) + z_2 \in \mathbb{C}[z_1, z_2]$

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(see formula (7) in Smyth's paper [17]). The quantity $M_0(P)$ was used by Mignotte [13] in estimating a remainder term in the Erdős-Turán theorem about the distribution of the roots of polynomials in sectors of the complex plane [11]. See also [8] and [19] for some improvements of Mignotte's result. In [1, Prop. 3.1] Amoroso has shown that

$$M_0(P) \geq e^{1/4} = 1.284025\dots \tag{1.3}$$

if $P \in \mathbb{Z}[z]$ is a product of cyclotomic polynomials.

Since

$$M_0(P) \geq M(P),$$

in connection to Lehmer's question it is natural to ask whether there is an absolute positive constant c_0 such that

$$M_0(P) \geq 1 + c_0 \tag{1.4}$$

for every nonconstant polynomial $P \in \mathbb{Z}[z]$ satisfying $P(0) \neq 0$.

We will show below that this is indeed the case and, moreover, that (1.4) holds not only for $P \in \mathbb{Z}[z]$, $P(0) \neq 0$, but also for every monic polynomial $P \in \mathbb{C}[z]$ satisfying $|P(0)| \geq 1$. The inequality (1.4) for nonconstant polynomials $P \in \mathbb{Z}[z]$, $P(0) \neq 0$, has been also obtained by Vesselin Dimitrov in 2018 (unpublished). In a correspondence between Dimitrov and the first named author we discussed that this result should also hold for complex polynomials. However, at that time we were unable to arrive to the proof of such a result. In this paper, we will not only show that (1.4) holds for each nonconstant monic $P \in \mathbb{C}[z]$ satisfying $|P(0)| \geq 1$ and find the optimal constant c_0, which turns out to be optimal for nonconstant polynomials $P \in \mathbb{Z}[x]$, $P(0) \neq 0$, too, but also prove a version of this inequality in a more general setting.

First, we introduce a more general family of measures for a polynomial $P \in \mathbb{C}[z]$ that includes the known cases mentioned above, namely,

$$M_u(P) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \max(u, \log|P(e^{i\theta})|)\,d\theta\right), \quad u \in \mathbb{R}. \tag{1.5}$$

It is clear that

$$\lim_{u\to-\infty} M_u(P) = M(P)$$

for any polynomial P. In a particular case, when $P \in \mathbb{C}[z]$ has no roots on the unit circle, we have $\mu(P) := \min_{|z|=1}|P(z)| > 0$. Then, $M_u(P) = M(P)$ for each real number $u \leq \log\mu(P)$.

Moreover, our results hold for even more general class of measures

$$M(\psi, P) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \psi(\log|P(e^{i\theta})|)\,d\theta\right), \tag{1.6}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is an increasing convex function with $\psi(-\infty) := \lim_{x\to-\infty}\psi(x)$. In particular, selecting $\psi(t) = \max(u, t)$ in (1.6) we get the measure M_u defined in (1.5). Since $\psi(\log|P(z)|)$ is a subharmonic function in $\mathbb{C}$ (cf. Theorems 2.2.2 and 2.6.3 of [16]), the submean inequality immediately gives the lower bound

$$M(\psi, P) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \psi(\log|P(e^{i\theta})|)\,d\theta\right) \geq \exp\left(\psi(\log|P(0)|)\right).$$

Our primary goal is to improve this trivial lower bound for monic polynomials with a prescribed value of $|P(0)|$.

Theorem 1.1. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be an increasing convex function. If $P(z)$ is a monic nonconstant polynomial in $\mathbb{C}[z]$ satisfying $|P(0)| = r > 0$, then*

$$M(\psi, P) \geq C(\psi, r) := \exp\left(\frac{1}{\pi}\int_0^\pi \psi\left(\log|e^{it} - r|\right)\,dt\right), \tag{1.7}$$

with equality holding in (1.7) *for all polynomials of the form $P(z) = z^n + c$, where $n \in \mathbb{N}$ and $|c| = r$. Moreover, if ψ is strictly convex, then equality holds in* (1.7) *only for polynomials $P(z) = z^n + c$ with $n \in \mathbb{N}$ and $|c| = r$.*

For the measure M_u introduced in (1.5), this result takes a more specific shape. We first observe that for $u > \log 2$, using in (1.5) the trivial inequality $\max(u, \log|P(e^{i\theta})|) \geq u$, we immediately obtain

$$M_u(P) \geq e^u, \tag{1.8}$$

which is sharp. Equality in (1.8) holds, for instance, for any $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \in \mathbb{C}[z]$ satisfying

$$|a_{n-1}| + \cdots + |a_0| \leq e^u - 1,$$

since then $\max(u, \log|P(e^{i\theta})|) = u$. For $u \leq \log 2$ we obtain the following:

Corollary 1.2. *If $P(z)$ is a monic nonconstant polynomial in $\mathbb{C}[z]$ satisfying $|P(0)| = 1$, then for each real number $u \leq \log 2$ we have*

$$M_u(P) \geq C_u := \exp\left(2\int_0^{x_u} \log(2\cos(\pi t))\,dt + u(1 - 2x_u)\right), \tag{1.9}$$

where $x_u := \frac{1}{\pi}\arccos\left(\frac{e^u}{2}\right)$. Equality holds in (1.9) *for all $P(z) = z^n + c$ with $n \in \mathbb{N}$ and $|c| = 1$.*

In Table 1 we give some values of the function C_u defined in (1.9).

TABLE 1. Some values of C_u

u	-5	-2	-1	0	0.5	$\log 2$
C_u	$1.002147\ldots$	$1.044031\ldots$	$1.126914\ldots$	$1.381356\ldots$	$1.733204\ldots$	2

Proposition 1.3. *The function C_u defined in* (1.9) *satisfies $C_u > 1$ for each $u \in \mathbb{R}$, and is increasing in the interval $(-\infty, \log 2]$.*

In particular, for $u = 0$ we obtain a sharp result for any monic $P \in \mathbb{C}[z]$ satisfying $|P(0)| = 1$, which improves (1.3). (Recall that (1.3) in [1] was proved only in a particular case when $P \in \mathbb{Z}[z]$ is a product of cyclotomic polynomials.) Moreover, our result establishes the optimal value of c_0 in (1.4). As in Boyd's paper [5] we set

$$I(\nu) := \int_0^\nu \log\left(2\cos\left(\frac{t}{2}\right)\right)\,dt. \tag{1.10}$$

(See also [6] and [10], where this integral appears in connection with the heights of polynomials.) Then, Corollary 1.2 with $u = 0$ implies the following:

Corollary 1.4. *If $P(z)$ is a monic nonconstant polynomial in $\mathbb{C}[z]$ satisfying $|P(0)| = 1$, then*

$$M_0(P) \ge \exp\left(\frac{1}{\pi} I\left(\frac{2\pi}{3}\right)\right) = M(1 + z_1 + z_2) = 1.381356\ldots. \tag{1.11}$$

Equality holds in (1.11) *for all $P(z) = z^n + c$ with $n \in \mathbb{N}$ and $|c| = 1$.*

Of course, by (1.7), the inequality (1.11) is true for any monic polynomial P in $\mathbb{C}[z]$ satisfying $|P(0)| \ge 1$, since

$$\max(0, \log|e^{it} - r|) \ge \max(0, \log|e^{it} - 1|)$$

for any $t \in [0, \pi]$ and $r = |P(0)| \ge 1$. Thus (1.4) holds for any monic polynomial $P \in \mathbb{C}[z]$ satisfying $|P(0)| \ge 1$ with

$$c_0 = C_0 - 1 = M(1 + z_1 + z_2) - 1 = 0.381356\ldots.$$

The result for $u = \frac{\log 2}{2}$ also gives some interesting constants:

Corollary 1.5. *If $P(z)$ is a monic nonconstant polynomial in $\mathbb{C}[z]$ satisfying $|P(0)| = 1$, then*

$$M_{\frac{\log 2}{2}}(P) \ge \exp\left(\frac{G}{\pi} + \frac{\log 2}{4}\right) = 1.591771\ldots, \tag{1.12}$$

where

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots = 0.915965\ldots$$

is Catalan's constant. Equality holds in (1.12) *for all $P(z) = z^n + c$ with $n \in \mathbb{N}$ and $|c| = 1$.*

2. Proofs

Our key tool is the Schur-Szegő composition (or convolution) of polynomials, which played a prominent role in the development of various polynomial inequalities. For a polynomial

$$\Lambda_n(z) = \sum_{k=0}^{n} \lambda_k \binom{n}{k} z^k \in \mathbb{C}[z], \tag{2.1}$$

we define the *Schur-Szegő composition* with another polynomial $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}[z]$ by

$$\Lambda P_n(z) := \sum_{k=0}^{n} \lambda_k a_k z^k. \tag{2.2}$$

If Λ_n is a fixed polynomial, then ΛP_n is a multiplier (or convolution) operator acting on a subspace of polynomials P_n. More information on the history and applications of this composition may be found in [2], [3], [9] and [15]. In particular, de Bruijn and Springer [9] proved a remarkable inequality that contains several well known inequalities about Mahler measure, see also a survey [14]. The method of de Bruijn and Springer was substantially developed by Arestov [2], who generalized it to a wide class of integral inequalities for polynomials.

We use the following result from [2], which is condensed from Theorems 4 and 5 there, and is adapted to our notation. We define a class of degree n complex polynomials $Z_n(\overline{\mathbb{D}})$ consisting of polynomials Λ_n as in (2.1) such that for any polynomial $P_n(z) = \sum_{k=0}^n a_k z^k \in$

$\mathbb{C}[z]$, $a_n \neq 0$, with all roots in the closed unit disk $\overline{\mathbb{D}}$, the polynomial ΛP_n as in (2.2) again has all its roots in $\overline{\mathbb{D}}$.

Theorem 2.1 (Arestov [2]). *Let $\psi : \mathbb{R} \to \mathbb{R}$ be an increasing convex function. If $P_n(z) = \sum_{k=0}^n a_k z^k \in \mathbb{C}[z]$, $a_n \neq 0$, $\Lambda_n \in Z_n(\overline{\mathbb{D}})$ and $|\lambda_n| = |\lambda_0| = 1$, then*

$$\int_0^{2\pi} \psi(\log |\Lambda P_n(e^{i\theta})|)\, d\theta \leq \int_0^{2\pi} \psi(\log |P_n(e^{i\theta})|)\, d\theta, \tag{2.3}$$

with equality in (2.3) *for $P_n(z) = a_n z^n + a_0$. Moreover, if ψ is strictly convex, then equality holds in* (2.3) *only for polynomials $P_n(z) = a_n z^n + a_0$.*

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We apply Theorem 2.1 by using the Schur-Szegő composition with $\Lambda_n(z) = z^n + 1$, so that $\lambda_n = \lambda_0 = 1$ and $\lambda_k = 0$ for $k = 1, \ldots, n-1$. We need to check that $\Lambda_n \in Z_n(\overline{\mathbb{D}})$. Consider any polynomial

$$P_n(z) = \sum_{k=0}^n a_k z^k = a_n \prod_{k=1}^n (z - z_k) \in \mathbb{C}[z],\ a_n \neq 0,$$

with all roots in $\overline{\mathbb{D}}$. By (2.2), the corresponding polynomial is $\Lambda P_n(z) = a_n z^n + a_0$. Since $|a_0|/|a_n| = \prod_{k=1}^n |z_k| \leq 1$, we conclude that ΛP_n has all its roots in $\overline{\mathbb{D}}$ too, as required, so $\Lambda_n(z) = z^n + 1$ is indeed in $Z_n(\overline{\mathbb{D}})$.

Therefore, for any polynomial $P(z) = z^n + a_{n-1} z^{n-1} \ldots + a_0 \in \mathbb{C}[z]$, from (1.6) and Theorem 2.1 we obtain

$$2\pi \log M(\psi, P) = \int_0^{2\pi} \psi(\log |P(e^{i\theta})|)\, d\theta \geq \int_0^{2\pi} \psi(\log |e^{in\theta} + a_0|)\, d\theta.$$

Recall that $|a_0| = r$ by our assumption, so we can write $a_0 = -re^{i\varphi}$ with some $\varphi \in \mathbb{R}$. Using periodicity and symmetry of the integral on the right hand side, by changing the variable $\theta = t/n$, we arrive at

$$\begin{aligned}\int_0^{2\pi} \psi(\log |e^{in\theta} + a_0|)\, d\theta &= \int_0^{2\pi} \psi(\log |e^{in\theta} - re^{i\varphi}|)\, d\theta = \int_0^{2\pi} \psi(\log |e^{in\theta} - r|)\, d\theta \\ &= 2n \int_0^{\pi/n} \psi(\log |e^{in\theta} - r|)\, d\theta = 2 \int_0^{\pi} \psi\left(\log |e^{it} - r|\right)\, dt,\end{aligned}$$

which implies (1.7). The case of equality in (1.7) is completely described by Theorem 2.1. □

We note that Theorem 2 of [4] is another result that could also be used in the proof of Theorem 1.1.

Proof of Corollary 1.2. We will apply Theorem 1.1 to the function $\psi(t) = \max(u, t)$ and to $r = 1$. By (1.5), (1.6), (1.7) and (1.9) we find that

$$\begin{aligned}\log M_u(P) &\geq \frac{1}{\pi} \int_0^{\pi} \max\left(u, \log |e^{it} - 1|\right)\, dt = \frac{1}{\pi} \int_0^{\pi} \max\left(u, \log\left(2 \sin\left(\frac{t}{2}\right)\right)\right)\, dt \\ &= \frac{1}{\pi} \int_0^{\pi} \max\left(u, \log\left(2 \cos\left(\frac{t}{2}\right)\right)\right)\, dt = 2 \int_0^{\frac{1}{2}} \max(u, \log(2\cos(\pi t)))\, dt.\end{aligned}$$

Now, we will show that

$$2\int_0^{\frac{1}{2}} \max(u, \log(2\cos(\pi t)))\,dt = 2\int_0^{x_u} \log(2\cos(\pi t))\,dt + u(1-2x_u). \tag{2.4}$$

To see that this is indeed the case we observe that the function $\phi(t) = \log(2\cos(\pi t))$ is decreasing in the interval $t \in [0, 1/2)$. For $x_u = \frac{1}{\pi}\arccos\left(\frac{e^u}{2}\right)$ and $u \le \log 2$ we clearly have

$$\phi(x_u) = \phi\left(\frac{1}{\pi}\arccos\left(\frac{e^u}{2}\right)\right) = \log\left(2\cdot\frac{e^u}{2}\right) = u.$$

Hence, $\max(u, \phi(t)) = \phi(t)$ for $t \in [0, x_u]$ and $\max(u, \phi(t)) = u$ for $t \in [x_u, \frac{1}{2}]$. This implies that the left hand side of (2.4) equals

$$2\int_0^{x_u} \log(2\cos(\pi t))\,dt + 2\int_{x_u}^{\frac{1}{2}} u dt = 2\int_0^{x_u} \log(2\cos(\pi t))\,dt + 2u\left(\frac{1}{2} - x_u\right),$$

which is $\log C_u$ by (1.9). This completes the proof of (2.4) and yields $\log M_u(P) \ge \log C_u$. □

Proof of Proposition 1.3. In the proof of Corollary 1.2 we have shown that

$$\log C_u = \frac{1}{\pi}\int_0^{\pi} \max\left(u, \log|e^{it} - 1|\right)\,dt. \tag{2.5}$$

Letting $u \to -\infty$ and using (1.1) with $P(z) = z - 1$ we obtain

$$\log(C_{-\infty}) = \frac{1}{\pi}\int_0^{\pi} \log|e^{it} - 1|\,dt = \frac{1}{2\pi}\int_0^{2\pi} \log|e^{it} - 1|\,dt = \log M(z-1) = 0,$$

so $C_{-\infty} = 1$. Furthermore, by (2.5), we see that $\log C_u$ is increasing in u. In particular, this implies $C_u > C_{-\infty} = 1$ for each $u \in \mathbb{R}$, and completes the proof of the proposition. □

Proof of Corollary 1.4. By Corollary 1.2 with $u = 0$ and (1.10), we find that $x_0 = \frac{1}{3}$ and

$$\log M_0(P) \ge \log C_0 = 2\int_0^{\frac{1}{3}} \log(2\cos(\pi t))\,dt = \frac{1}{\pi}\int_0^{\frac{2\pi}{3}} \log\left(2\cos\left(\frac{t}{2}\right)\right)\,dt = \frac{1}{\pi} I\left(\frac{2\pi}{3}\right).$$

By [5] (see several expressions for the number β in [5]), the right hand side is also equal to the logarithm of the Mahler measure of the polynomial $1 + z_1 + z_2$ in two variables. The numerical value $M(1 + z_1 + z_2) = 1.381356444518\ldots$ was also calculated by Boyd [5]. □

Proof of Corollary 1.5. By Corollary 1.2 with $u = \frac{\log 2}{2}$ and (1.10), we now have $x_{\frac{\log 2}{2}} = \frac{1}{4}$ and

$$\log M_{\frac{\log 2}{2}}(P) \ge \log C_{\frac{\log 2}{2}} = 2\int_0^{\frac{1}{4}} \log(2\cos(\pi t))\,dt + \frac{\log 2}{4} = \frac{1}{\pi} I\left(\frac{\pi}{2}\right) + \frac{\log 2}{2}.$$

This implies the corollary, due to $I\left(\frac{\pi}{2}\right) = G$ (see [5]). □

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Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania
Email address: arturas.dubickas@mif.vu.lt

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, U.S.A.
Email address: igor@math.okstate.edu